## PHYSICS DEPARTMENT

## Year 12

## Introduction to Mathematical Methods in Physics

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## SI UNITS

The National Physical Laboratory (NPL) which is responsible for metrology, the science of measurement and units, in the UK describes the SI system as:

Formally agreed by the 11th General Conference on Weights and Measures (CGPM) in 1960 , the SI is at the centre of all modern science and technology. The definition and realisation of the base and derived units is an active research topic for metrologists with more precise methods being introduced as they become available. There are two classes of units in the SI: base units and derived units. The base units provide the reference used to define all the measurement units of the system, whilst the derived units are products of base units and are used as measures of derived quantities:

- The seven SI base units, which comprise:

$$
\begin{array}{ll}
\circ & \text { The ampere (A) - unit of measurement of electric current } \\
\circ & \text { The kilogram (kg) - unit of measurement of mass } \\
\circ & \text { The metre (m) - unit of measurement of length } \\
\circ & \text { The second (s) - unit of measurement of time } \\
\circ & \text { The kelvin }(\mathrm{K}) \text { - unit of measurement of thermodynamic temperature } \\
\circ & \text { The mole (mol) - unit of measurement of amount of substance } \\
\circ & \text { The candela (cd) - unit of measurement of luminous intensity }
\end{array}
$$

The seven bases units have internationally agreed values, all other units are derived from these seven. The NPL gives these examples of derived units:

- Examples of SI derived units

| Examples of SI derived units expressed in terms of base units |  |  |
| :--- | :--- | :--- |
| Derived Quantity |  | SI derived unit |
|  | Name | Symbol |
| area | square metre | $\mathbf{m}^{2}$ |
| volume | cubic metre | $\mathbf{m}^{3}$ |
| speed, velocity | metre per second | $\mathbf{m s}^{-1}$ |
| acceleration | metre per second squared | $\mathbf{m}^{-1}$ |
| wavenumber | 1 per metre | $\mathbf{k g m}^{-3}$ |
| density, mass density | kilogram per cubic metre | $\mathbf{m}^{3} \mathbf{k g}^{-1}$ |
| specific volume | cubic metre per kilogram |  |


| current density | ampere per square metre | $\mathrm{Am}^{-2}$ |
| :--- | :--- | :--- |
| magnetic field strength | ampere per metre | $\mathrm{Am}^{-1}$ |
| concentration <br> (of amount of substance) | mole per cubic metre | $\mathbf{m o l m}^{-3}$ |
| luminance | candela per square metre | $\mathbf{c d m}^{-2}$ |
| refractive index | (the number) one | $\mathbf{1}^{(\mathrm{a})}$ |
|  |  |  |

${ }^{(a)}$ The symbol '1' is generally omitted in combination with a numerical value.

22 SI derived units have been given special names and symbols. The following list is from the NIST, the USA's version of the NPL:

| Derived quantity | SI derived unit |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  | Name | Symbol | Expression in terms of other SI units | Expression in terms of SI base units |
| plane angle | radian | rad | - | $\mathbf{m} \cdot \mathrm{m}^{-1}=1$ |
| solid angle | steradian | sr | - | $\mathrm{m}^{2} \cdot \mathrm{~m}^{-2}=1$ |
| frequency | hertz | Hz | - | $\mathrm{s}^{-1}$ |
| force | newton | N | - | $\mathrm{m} \cdot \mathrm{kg} \cdot \mathrm{s}^{-2}$ |
| pressure, stress | pascal | Pa | $\mathrm{N} / \mathrm{m}^{2}$ | $\mathrm{m}^{-1} \cdot \mathrm{~kg} \cdot \mathrm{~s}^{-2}$ |
| energy, work, quantity of heat | joule | J | $\mathbf{N} \cdot \mathrm{m}$ | $\mathrm{m}^{2} \cdot \mathrm{~kg} \cdot \mathrm{~s}^{-2}$ |
| power, radiant flux | watt | W | J/s | $\mathrm{m}^{2} \cdot \mathrm{~kg} \cdot \mathrm{~s}^{-3}$ |
| electric charge, quantity of electricity | coulomb | C | - | $s \cdot A$ |
| electric potential difference, electromotive force | volt | V | W/A | $\mathrm{m}^{2} \cdot \mathrm{~kg} \cdot \mathrm{~s}^{-3} \cdot \mathrm{~A}^{-1}$ |
| capacitance | farad | F | C/V | $\mathrm{m}^{-2} \cdot \mathrm{~kg}^{-1} \cdot \mathrm{~s}^{4} \cdot \mathrm{~A}^{2}$ |
| electric resistance | ohm | $\Omega$ | V/A | $\mathrm{m}^{2} \cdot \mathrm{~kg} \cdot \mathrm{~s}^{-3} \cdot \mathrm{~A}^{-2}$ |


| electric conductance | siemens | S | A/V | $\mathrm{m}^{-2} \cdot \mathrm{~kg}^{-1} \cdot \mathrm{~s}^{3} \cdot \mathrm{~A}^{2}$ |
| :---: | :---: | :---: | :---: | :---: |
| magnetic flux | weber | Wb | V•s | $\mathrm{m}^{2} \cdot \mathrm{~kg} \cdot \mathrm{~s}^{-2} \cdot \mathrm{~A}^{-1}$ |
| magnetic flux density | tesla | T | $\mathbf{W b} / \mathrm{m}^{2}$ | $\mathrm{kg} \cdot \mathrm{s}^{-2} \cdot \mathrm{~A}^{-1}$ |
| inductance | henry | H | Wb/A | $\mathrm{m}^{2} \cdot \mathrm{~kg} \cdot \mathrm{~s}^{-2} \cdot \mathrm{~A}^{-2}$ |
| Celsius temperature | degree <br> Celsius | ${ }^{\circ} \mathrm{C}$ | - | K |
| luminous flux | lumen | 1 m | cd $\cdot \mathbf{s r}{ }^{(c)}$ | $\mathrm{m}^{2} \cdot \mathrm{~m}^{-2} \cdot \mathbf{c d}=\mathrm{cd}$ |
| illuminance | lux | 1x | $1 \mathrm{~m} / \mathrm{m}^{2}$ | $\mathrm{m}^{2} \cdot \mathrm{~m}^{-4} \cdot \mathrm{~cd}=\mathrm{m}^{-2} \cdot \mathbf{c d}$ |
| activity (of a radionuclide) | becquerel | Bq | - | $\mathrm{s}^{-1}$ |
| absorbed dose, specific energy (imparted) | gray | Gy | J/kg | $\mathrm{m}^{2} \cdot \mathrm{~s}^{-2}$ |
| dose equivalent | sievert | Sv | $\mathrm{J} / \mathrm{kg}$ | $\mathrm{m}^{2} \cdot \mathrm{~s}^{-2}$ |
| catalytic activity | katal | kat |  | $\mathrm{s}^{-1} \cdot \mathrm{~mol}$ |

Radians are a measure of angle, used instead of degrees. We will be using radians in the course:


One radian is the angle subtended by two radii at the centre of a circle where the arc length equals the radius of that circle. Because of the formula for the circumference of a circle, a complete circle or $360^{\circ}$ is $2 \pi$ radians, so a right angle is $\pi / 2$ radians. While you would be crazy to do a measurement in radians, we always use degrees, the fact that radians arise from the geometry rather than arbitrarily choosing 360 to be the number of divisions in a circle gives useful mathematical consequences, the most important for us being the "small angle approximation".

There is one more unit of angle, the gradian, chosen such that a right angle is 100 gradians. This fits better with the metric system, but we do not use it. Be careful, however, because it is an option on your calculator.
$\underline{\text { SI prefixes are used to form decimal multiples and submultiples of SI units. }}$
They should be used to avoid very large or very small numeric values.
The prefix attaches directly to the name of a unit, and a prefix symbol attaches directly to the symbol for a unit.

| Multiplying Factor | SI Prefix | Scientific Notation |
| :---: | :---: | :---: |
| 1000000000000000000000000 | yotta (Y) | $10^{24}$ |
| 1000000000000000000000 | zetta (Z) | $10^{21}$ |
| 1000000000000000000 | exa (E) | $10^{18}$ |
| 1000000000000000 | peta (P) | $10^{15}$ |
| 1000000000000 | tera (T) | $10^{12}$ |
| 1000000000 | giga (G) | $10^{9}$ |
| 1000000 | mega (M) | $10^{6}$ |
| 1000 | kilo (k) | $10^{3}$ |
| 0.001 | milli (m) | $10^{-3}$ |
| 0.000001 | micro ( $\mu$ ) | $10^{-6}$ |
| 0.000000001 | nano (n) | $10^{-9}$ |
| 0.000000000001 | pico (p) | $10^{-12}$ |
| 0.000000000000001 | femto (f) | $10^{-15}$ |
| 0.000000000000000001 | atto (a) | $10^{-18}$ |
| 0.000000000000000000001 | zepto (z) | $10^{-21}$ |
| 0.000000000000000000000001 | yocto (y) | $10^{-24}$ |

button on your calculator puts numbers into an "engineering form". This is standard form, but where the index on the ten must be a multiple of three, as opposed to the first number having to be between one and ten. This allows easy conversion to units with prefixes.

## SOME INTERNATIONALLY RECOGNISED NON-SI UNITS

The NPL says:

There are certain units, which are accepted for use with the SI. It includes units which are in continuous everyday use, in particular the traditional units of time and of angle, together with a few other units which have assumed increasing technical importance.
There are also units which are currently accepted for use with the SI to satisfy the needs of commercial, legal and specialist scientific interests or are important for the interpretation of older texts.

| Non-SI units accepted for use with the International System |  |  |
| :---: | :---: | :---: |
| Name | Symbol | Value in SI Units |
| minute | min | $1 \mathrm{~min}=60 \mathrm{~s}$ |
| hour | h | $\mathbf{1} \mathrm{h}=60 \mathrm{~min}=3600 \mathrm{~s}$ |
| day | d | $1 \mathrm{~d}=24 \mathrm{~h}=86400 \mathrm{~s}$ |
| degree of arc | - | $1^{\circ}=(\pi / 180) \mathrm{rad}$ |
| minute of arc | ' | $1^{\prime}=(1 / 60)^{\circ}=(\pi / 10800) \mathrm{rad}$ |
| second of arc | " | $1^{\prime \prime}=(1 / 60)^{\prime}=(\pi / 648000) \mathrm{rad}$ |
| litre | 1, L | $11=1 \mathrm{dm}^{3}=10^{-3} \mathrm{~m}^{\mathbf{3}}$ |
| tonne | t | $1 \mathrm{t}=10^{3} \mathrm{~kg}$ |

There are three more units that we might encounter which are not SI , all three are units of astronomical distance:

| Astronomical Unit | AU | Mean Earth Sun distance | $1.50 \times 10^{11} \mathrm{~m}$ |
| :--- | :--- | :--- | :--- |
| Light Year | ly | Distance travelled by light in a year | $9.46 \times 10^{15} \mathrm{~m}$ |
| Parsec | pc | Distance at which parallax is an arcsecond | $3.09 \times 10^{16} \mathrm{~m}$ |

A now largely obsolete unit, the Angstrom, may come up in the course, or in your reading around. Symbol $\AA$, one angstrom is $10^{-10} \mathrm{~m}$, which is the same order of magnitude as the atomic spacings in a crystal and so is favored by crystallographers.

## SYMBOLS FOR VARIABLES AND THE GREEK ALPHABET

Not only are there symbols for the SI units and their prefixes there are also symbols commonly used within equations to represent physical quantities. There are so many physical quantities which could need representing that the Greek Alphabet is used to supplement the English Alphabet (even then symbols end up being reused). It is therefore important that you can recognize and draw at least the important ones (lower case; alpha, beta, gamma, delta, epsilon, theta, lambda, mu, pi, rho, omega, and upper case; delta, sigma, omega).


Physical quantities or variables are distinguished from units in printed text by the symbols for the variables being italicised. For example, in the unusual case of P.D. where the unit symbol and the usual variable symbol are both capital vee, 2 V would represent twice the variable, 2 V is a value of two volts. (The current SI convention is for there to be a space between the value and the unit - Dr Cooke is not in that habit and so the proper convention may not be followed in this document.)

The Greek Alphabet (usually upper case) is also used to name subatomic particles. Particles symbolised by upper case delta, kappa, lambda, xi, pi and omega, as well as lower case mu and tau, will appear on the course.

## EXERCISE

Young's Modulus, symbol $E$, has the same dimensions (see 1) as Pressure and so its units are Pascals. Young's Modulus charaterises the stretch or compression of a material under an applied load. A formula for it is:

$$
E=\frac{F l}{A \Delta l}
$$

Where $F$ is the tension applied to a wire, I the original length of the wire, $A$ its cross sectional area, and $\Delta /$ the increase in the wire's length due to the tension (standard SI units $\mathrm{N}, \mathrm{m}$ and $\mathrm{m}^{2}$ must be used).

You already know the wave equation:

$$
c=\lambda f
$$

Where, in these questions, $c$ is the speed of light in a vacuum $\left(3.00 \times 10^{8} \mathrm{~ms}^{-1}\right), \lambda$ is the wavelength and $f$ the wave frequency.

Using the two equations given above, convert into the correct base units, calculate and give the answer with an appropriate unit multiplier:

1. Find the frequency of a radio wave with a wavelength of 1750 mm
2. Find the wavelength of a radio wave with a 22.5 kHz frequency
3. Find the frequency of an x-ray that has a one point two three angstrom wavelength
4. Find the frequency of infrared light with an $1.05 \mu \mathrm{~m}$ wavelength
5. Find the wavelength of red light with a frequency of 442 THz
6. Find the Young's Modulus of Copper if a copper wire with a cross-sectional area of $3.20 \times 10^{-6} \mathrm{~m}^{2}\left(\times 10^{-6} \mathrm{~m}^{2} \text { could be written as } \mu\left(\mathrm{m}^{2}\right) \text { or even ( } \mathrm{mm}\right)^{2}$, but isn't to avoid confusion) and originally 107.0 cm long extends by $126 \mu \mathrm{~m}$ under the application of 42 N ?
7. How much would a hardened steel plate extend if it has a Young's Modulus of 196GPa, a cross sectional area of $1.5 \times 10^{-2} \mathrm{~m}^{2}$ and a length of 750 mm , if 700 kN were applied?
8. Plastic cord has a Young's Modulus of 816 MPa and a radius of 0.75 mm , how much load is necessary to extend 2.00 m by 3 mm ?

## EXERCISE

Without going back to look until you have tried to remember for yourself add the prefix names and symbols:

| 1000000000000 |  | $10^{12}$ |
| :--- | :--- | :---: |
| 1000000000 |  | $10^{9}$ |
| 1000000 |  | $10^{6}$ |
| 1000 |  | $10^{3}$ |
| 0.001 |  | $10^{-3}$ |
| 0.000001 |  | $10^{-6}$ |
| 0.000000001 |  | $10^{-9}$ |
| 0.000000000001 |  | $10^{-12}$ |

Write out the greek letters:
Lower case
Upper case

Alpha $\qquad$ Delta. $\qquad$

Beta.
Lambda. $\qquad$

## Gamma

Xi. $\qquad$

Delta. $\qquad$ Pi. $\qquad$

Epsilon $\qquad$ Sigma $\qquad$

Theta. $\qquad$ Omega $\qquad$

Lambda $\qquad$

Mu. $\qquad$

Rho $\qquad$

Omega $\qquad$

## LARGE AND SMALL NUMBERS

Although we encouraged you to use them, standard form and engineering form were not really necessary for GCSE Physics because the numbers involved were usually on a "human" scale. This is absolutely not the case in A level Physics, we will be dealing with microscopic entities like the electron and macroscopic ones like the Galaxy. Once the scale gets beyond about $10^{ \pm 15}$ scientists often give up on the multipliers from the section before and just use standard form. In your course we will tend to use standard form beyond $10^{ \pm 12}$.

A LARGE AND SMALL CALCULATIONS EXERCISE. THE SUN.

On average a square metre on the earth receives 340 W in radiant power from the sun. However the earth is a three dimensional body, but the easiest way to think about the solar radiation arriving is as a circle whose radius is that of the earth $\left(R_{E}\right)$ cut from the sphere that the sun's whole radiance is being spread out over at the earth's distance from the sun.


Because the area of that circle is $\pi R_{E}{ }^{2}$, but the surface area of the earth (a sphere) is $4 \pi R_{E}{ }^{2}$, the power per square metre arriving from the sun at the sun-earth distance $\left(R_{S-E}\right)$ is four times the average arriving at the curve of the earth's surface. Use this information to show how you would calculate the total solar power (irradiance) arriving at the earth $\left(I_{E}\right)$ and then the total energy being output by the sun $\left(I_{S}\right)$.
$I_{E}=$

## $I_{s}=$

You should obtain values of $I_{E}=1.73 \times 10^{17} \mathrm{~W}$ and $I_{S}=3.85 \times 10^{26} \mathrm{~W}$
The sun creates energy by nuclear fusion - i.e. fusing four hydrogen nuclei (four protons) into a single helium nucleus.
The mass of a proton is $1.67262 \times 10^{-27} \mathrm{~kg}$
The mass of a helium nucleus is $6.64466 \times 10^{-27} \mathrm{~kg}$
There is a difference in the starting mass of the protons and the final mass of the nucleus, this is called the mass defect. Mass defect relates to the energy released via Einstein's famous formula:

$$
E=m c^{2}
$$

Where $c$ is the speed of light $2.99820 \times 10^{8} \mathrm{~ms}^{-1}$.
Use the information above to calculate the energy released by the creation of one helium nucleus ( $E_{\text {fus }}$ ).
$E_{\text {fus }}=$
Calculate how many helium nuclei ( $N_{H e}$ ) must be forming per second to produce the output power of the sun (assuming all the energy goes into light).
$N_{H e}=$
If the mass of an electron is $9.10938 \times 10^{-31} \mathrm{~kg}$, how many kilograms of hydrogen are consumed every second ( $m_{H}$ )?
$m_{H}=$
What effect did including the electron mass have on this calculation and why?
$\qquad$
$\qquad$
$\qquad$

Rewrite the mass of hydrogen fused per second as tonnes of hydrogen per day ( $m_{H_{p . d}} 1$ tonne $=10^{3} \mathrm{~kg}$ )
$\boldsymbol{m}_{H_{\text {p.d. }}}=$
If the mass of the sun is $1.99 \times 10^{30} \mathrm{~kg}$ and it started off almost completely made from hydrogen what is your estimate for the lifetime of the sun $\left(T_{s}\right)$ ?
$T_{s}=$
Show how you convert this value into years ( $T_{S_{y r}}$ )
$\boldsymbol{T}_{S_{y r}}=$
You should get a value of the order of 100 billion years, if you did not go back and find your mistake(s).
The actual estimate for the lifetime of the sun in its current phase is 9 billion years, how might you explain this discrepancy?
$\qquad$
$\qquad$
$\qquad$

## DIMENSIONAL ANALYSIS

For an equation in Physics to be true the values obtained from each side must not only be the same, but their units must be equivalent. A method of formally testing this is known as dimensional analysis.

In dimensional analysis we are only concerned with the nature of the dimension i.e. its quality not its quantity. The following common abbreviations are used:
length $=\mathrm{L}$
mass $=\mathrm{M}$
time $=T$
charge = Q
temperature $=\Theta$
Newton's Second Law tells us that force is proportional to rate of change of momentum. Momentum is mass times velocity, velocity is length divided by time, and rate means divided by time again and so by using the abbreviations above the dimensions of force are:
$\mathrm{MLT}^{-2}$
Work done or energy is force times distance so its dimensions are:
$M L^{2} T^{-2}$
Example: Determine the dimensions of Specific Heat Capacity
The equation for Energy and Specific Heat Capacity can be rearranged to;

$$
C=\frac{E}{m \Delta T}
$$

We know the dimensions of energy, adding in the other two gives:

$$
\mathrm{ML}^{2} \mathrm{~T}^{-2} / \mathrm{M} \Theta
$$

The dimensions of Specific Heat Capacity are therefore:

$$
\mathrm{L}^{2} \mathrm{~T}^{-2} \Theta^{-1}
$$

DIMENSIONAL ANALYSIS EXERCISE

1. Using Dimensional Analysis show that the two possible units for the acceleration due to gravity, $\mathrm{g}, \mathrm{N} / \mathrm{kg}$ and $\mathrm{ms}^{-2}$ are equivalent.
2. Potential Energy in a gravitational field is given by the equation:

$$
E=\frac{G m_{1} m_{2}}{r}
$$

Where $m_{1}$ and $m_{2}$ are masses and $r$ is the distance between them.

Find the dimensions of G; Newton's Universal Gravitational Constant.
3. The equation for the time period of a mass $m$ bouncing on a spring is given by:

$$
T=2 \pi \sqrt{\frac{m}{k}}
$$

Where $k$ is the same spring constant we met at GCSE ( $F=k x$ ).
Show that the dimensions of both sides of this equation are time.

## VECTORS

We met the idea of vectors at GCSE. A physical vector quantity is one that has both mass and direction. Examples are force, velocity and momentum whereas examples of scalar quantities might be mass or energy.

## VECTOR DIAGRAMS

At GCSE we also met the idea of graphical vector addition of two forces. There are two possible methods draw a parallelogram, or move the vectors so that the end of one makes the beginning of the next:

Example: A point mass has two forces acting upon it as shown, construct an accurate force diagram show the magnitude and direction of the resultant of the two forces.

Both methods are easier to construct with compasses, but with the parallelogram method a pair of compasses makes a protractor unnecessary.


Using the compasses an arc the radius of one vector is drawn centred on the end of the other, when this is repeated the other way around the crossing of the two arcs marks the end of the resultant.

The two vector method is conceptually harder, vector addition occurs when the vectors are lined up to flow from one to another, so you have to imagine picking up one vector and placing it on the end of the other and then constructing this accurately.


You can see that as with normal addition it does not matter which vector is first and which is second; doing the construction in the opposite order would result in the other two sides of the parallelogram above being drawn, $R$ would not move.

The advantage to the second method is that it is easier to see how more vectors would be included.
Example 2: if weight were included in the example above:
Adding in the weight vector to our existing vector diagram is relatively simple:


It can be seen that the resultant force is now sideways and so the effect of the three forces would be to accelerate the point mass to the right.

If the all of the vectors in a vector diagram form a closed shape, for example a triangle or quadrilateral, then there is no room to draw in a resultant. If there is no resultant then the forces are balanced. If the forces acting on a point are balanced then the point is in equilibrium. (If the forces are acting across an extended body like a beam then there might still be an overall moment.)

Example 3: A weight in equilibrium is hanging from a newton meter, but is being pulled sideways by a horizontal piece of string tied at the same point of attachment as the meter. The newton meter makes an angle of $30^{\circ}$ with the horizontal. If the newton meter reads 4 N construct a force diagram to determine the magnitude of the weight and the tension in the string.

The first thing to do with a complex question is to always do a drawing:


And mark on the forces and angles (properly weight should act from the centre of a body, but you know from GCSE that its line of action must be through the support or pivot point):


Accurately constructing a scale diagram seems a tall order because we only know one force. However, we do know there are three forces in equilibrium, therefore the three will make a closed figure, which has to be a triangle. Additionally two of the forces act at right angles to one another so our force diagram will be a right angled triangle:


Accurately construct a scaled version of the 4 N force, the weight must be vertically downwards, and the tension must complete the shape horizontally. The forces can now be measured and calculated from your scale to give a weight of 2 N and a tension of 3.5 N .

Vector diagrams for vector addition represent the magnitude (by length) and the direction of the vector, but are not just scale diagrams of the problem because the vectors may be placed relative to one another (end to end) rather than being placed as they are in real space. The force diagrams are a representation of a vector space, not real space.

## EQUILIBRIUM BY VECTOR ADDITION EXERCISE

A 2 N weight is supported in equilibrium by three elastic strings tied off at the same point above the weight's centre of gravity. One string is pulling the weight to the right horizontally with a force of 1.2 N . One string is pulling to the left and down with a force of 2.7 N , it makes an angle of $15^{\circ}$ to the horizontal. The final string is mostly pulling the weight upwards, construct a scale diagram to find the tension in the last string and to find the angle it makes to the vertical.

## VECTOR SUBTRACTION

As with normal subtraction, vector subtraction finds the difference between two vectors.
Example: Consider a picture held up by a pair of angled strings, if the two strings are at $20^{\circ}$ to the horizontal and the tension in each is twelve point five newtons, find the difference between the two tensions.


Vector Addition would allow us to find $W$, either by finding the resultant and taking $W$ to be its opposite or by finding the missing force in the closed figure (hopefully you can see that the diagrams for these two would only differ by the direction of the final arrow.)

Vector subtraction is done by placing the vectors so that they do not flow from one to another, but both move out from the same starting position


Unfortunately, the fact that the difference between $T_{1}$ and $T_{2}$ is a vector 23.5 N rightwards does not actually tell us anything useful - you could have worked out that the two differed only by their portions acting horizontally just by looking at them, and so you will not see questions like this.

Where vector subtraction is more useful is with a different vector quantity; not force but velocity.

## RELATIVE VELOCITY

The difference between two velocity vectors gives the relative velocities of the two. The idea of relative velocity is easiest to understand in a straight line:

Example: Consider two cars travelling eastwards along a road. Car 1 starts off 2 km in front of Car 2 and is travelling at $22 \mathrm{~ms}^{-1}$, Car 2 is travelling at $36 \mathrm{~ms}^{-1}$, how long will it take Car 2 to catch Car 1?

This being Physics we start with a drawing:


The difference in Car 1's velocity compared to Car 2's is Car 1's velocity minus Car 2's, which is obviously 14 ms ${ }^{1}$, but graphically this is:

$14 \mathrm{~ms}^{-1}$ eastwards is the relative velocity of Car 1 compared to Car 2 or the change that an observer in Car 2 would see looking out at Car 1 (the observer is stationary within their own frame of reference).

The maths is straightforward; 2 km will be covered in 143 s at a closing velocity of $14 \mathrm{~ms}^{-1}$.

Example 2: Two speedboats set out from the same point. One is travelling at $8 \mathrm{~ms}^{-1}$ northeast, the second is travelling at $15 \mathrm{~ms}^{-1}$ ten degrees east of south, using a graphical method determine how long will it take for them to be 5 km apart.


The magnitude of the difference between the two speed vectors can be found from an accurate scale diagram; which for once resembles the real space diagram:


An observer in the blue boat will see the red boat moving away just west of south at a speed (speed is the correct term because it is the magnitude) of $21.6 \mathrm{~ms}^{-1}$. Therefore it will take three minutes fifty one and a half seconds for the two boats to be 5 km apart (5000 $\div 21.6$ ).

## RELATIVE VELOCITY EXERCISE

Three planes take off from Jersey at roughly the same time, they immediately change to new bearings. (You should remember from maths that bearings are three figure numbers, which represent the degrees clockwise from north). The commercial turboprop travels east (090) at $580 \mathrm{~km} / \mathrm{h}$, the small jet's bearing is 340 and it travels at $620 \mathrm{~km} / \mathrm{h}$, meanwhile the private single-engined plane heads towards Brittany on a bearing of 250 at $200 \mathrm{~km} / \mathrm{h}$. Use a graphical method to determine which pair of planes is furthest apart after half an hour of flight, and determine how much further apart they are than the next furthest pair.

Extra Distance over Next Furthest Pair

## RESOLVING INTO ORTHOGONAL VECTORS

Of course graphical methods are all well and good, and they help you to establish a model in your head of what is going on, but relying on measuring a scale drawing is a bit too close to Biology for comfort. We therefore usually use a purely mathematical method.

The method used is to split (resolve) each single vector into the sum of two vectors in directions which are perpendicular to one another.

To use an example with easy numbers; a vector five units long acting rightwards at an angle of $36.9^{\circ}$ above the horizontal can be written as the sum of a horizontal vector acting rightwards four units long and a vertical vector acting upwards three units long (this sentence should make sense when you look at the diagram).


If we call rightwards the $x$-direction and upwards the $y$-direction, choices we would make clear to the examiner by sketching them on the diagram as above, then our resultant five unit vector $\boldsymbol{R}$ can be written as:

$$
R=4 \hat{x}+3 \hat{y}
$$

Where $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ are vectors in the x and y directions respectively, which are each one unit in length.
Writing out the component vectors this formerly is more Maths than Physics at our level. The important thing from an A level Physics perspective is that by choosing perpendicular directions processes occurring in the $x$-direction are independent of processes occurring in the $y$-direction. There is no portion of the $x$ vector that impinges on the $y$ direction and vice versa, the directions are orthogonal.

All of which means that if we want to combine vectors, we resolve them into two orthogonal directions, combine the $x$ 's, combine the $y$ 's and then if necessary reassemble them into a final vector.

This is an abstract idea, but a very powerful one which occurs again and again throughout Physics.

## SINE AND COSINE

You will have met the trigonometric functions in Maths. The important idea about them that is not always strongly emphasised in Maths is that sine and cosine are the projections in two perpendicular directions of a unit vector sweeping out a circle as the angle $\vartheta$ between the vector and the x axis increases.


This is illustrated in the diagram above.
Note that the angles in the diagram are measured in radians, where $\pi$ radians is the equivalent of $180^{\circ}$. The fact that sine has to be rotated by $\pi / 2$ radians to obtain cosine tells us that the sine and cosine functions are " $\pi / 2$ out of phase" which is an idea that will crop up in the waves course.

The conceptual relationship between sine and cosine, waves and circular motion will occur often during the course.
The relationship between sine and cosine and resolving vectors should also become clear from the diagram. If you consider the radius sweeping around the circle to be a vector that we want to resolve then our unit vector has the components:
$\hat{x} \cdot \cos \vartheta$ and $\hat{y} \cdot \sin \vartheta$

Note that during its rotation around the circle our unit vector sweeps out a scaled version of every possible right angled triangle.

If the vector we wish to resolve is not a unit vector, but has a length $L$, then the vector $L$ can be resolved as:

$$
L=\hat{x} \cdot L \cdot \cos \vartheta+\hat{y} \cdot L \cdot \sin \vartheta
$$

Taking sine (the side opposite the angle) and cosine (the side next to the angle) will always give us two perpendicular components of a vector.

## RESOLVING VECTORS

Example: Find the $x$ and $y$ components of a force that has a magnitude of 35 N and is acting at an angle of $40^{\circ}$ below the $x$ axis.

As always we start with the diagram:


The $x$ component is adjacent to the angle so we use cos:
$=35 \cos 40=26.8 \mathrm{~N}$
The y component is therefore sin:
$=35 \sin 40=22.5 \mathrm{~N}$
But the $y$ component is downwards so the answer is actually -22.5 N

In the example above, had we used $320^{\circ}$, the full angle swept out by the rotation of our vector around to that point, the negative sign would have arisen automatically from the maths, i.e. $35 \cos 320=26.8 \mathrm{~N}$ and $35 \sin 320=-22.5 \mathrm{~N}$, but usually we just remember to think about the negative signs.

Example 2: Find the horizontal and vertical components of a force that has a magnitude of 80 kN and is acting at an angle of $\pi / 6$ radians to the right of vertical.


There are a number of approaches here, but the simplest is to remember pi radians is 180 degrees, so the angle must be $30^{\circ}$. Similarly we could use $60^{\circ}$ because that is the angle that the vector would sweep out, as above, if we think of vertical as $y$ and right as $x$. You could even use the sign rule if you were that way inclined. However, easiest is something like:

The horizontal component is opposite the angle so use sin:

$$
=80 \sin 30=40 \mathrm{kN} \text { rightwards }
$$

The vertical component is therefore cos:

$$
=80 \cos 30=69.3 \mathrm{kN} \text { upwards }
$$

In this second example the directions are not $x$ and $y$, but vertical and horizontal, this is typical in mechanics problems, but the directions could be the compass points, or in calculations involving slopes parallel with the slope and perpendicular to the slope.

## VECTOR CALCULATION BY RESOLVING INTO COMPONENTS

Vector addition or subtraction once the resolving is complete is straightforward, simply add or subtract the perpendicular components separately.

Example: Redo Example 2 from the relative velocity section, by calculation rather than graphically.

The components for the blue boat are:

$8 \sin 45=5.657 \mathrm{~ms}^{-1}$ East
and
$8 \sin 45=5.657 \mathrm{~ms}^{-1}$ North

For the red boat:
$16 \sin 10=2.778 \mathrm{~ms}^{-1}$ East
and
$=16 \cos 10=15.757 \mathrm{~ms}^{-1}$ South

South is the negative of north so we will assign south a negative sign, subtracting the blue boat's vector from the red's to find the relative velocity of the red boat as seen by the blue gives:

North: $\quad-15.757-5.657=-21.41 \mathrm{~ms}^{-1}$
East: $\quad 2.778-5.657=-2.879 \mathrm{~ms}^{-1}$

Both components are negative telling us that the directions are actually South and West.
We now need to recombine these into a single vector:
$2.879 \mathrm{~ms}^{-1}$


The final vector's magnitude is given using Pythagoras (the double lines - sometimes single lines - mean vector magnitude):
$\|v\|=\sqrt{2.879^{2}+21.41^{2}}=21.6 \mathrm{~ms}^{-1}$
For the direction we use tan:
$\tan ^{-1}\left(\frac{2.879}{21.41}\right)=7.66^{\circ}$ west of south
The question asked for the time to a separation of 5 km , as before this is $5000 \div 21.6=231 \mathrm{~s}$

## EXERCISE ${ }^{1}$

1. By calculation find the $X$ and $Y$ components of the following:
A. $35 \mathrm{~m} / \mathrm{s}$ at $57^{\circ}$ from the x -axis.
B. $12 \mathrm{~m} / \mathrm{s}$ at $34^{\circ} \mathrm{S}$ of W
C. $8 \mathrm{~m} / \mathrm{s}$ South
D. $20 \mathrm{~m} / \mathrm{s} 275^{\circ}$ from the x -axis

[^0]2. Find the resultant vector (mag and dir) given the following information:
A. $A_{x}=5.7, A_{y}=3.4$
B. $B_{x}=-10, B_{y}=-3$
C. $\mathrm{C}_{\mathrm{x}}=12, \mathrm{C}_{\mathrm{y}}=-20$
D. $-30 \mathbf{i}+27 \mathbf{j}$
E. $48 \mathbf{i}-12 \mathbf{j}$
3. How fast is the plane travelling and what angle is it making to the horizontal?

4. What is the force on the fishing line, given that the horizontal component is 50 N ?

5. Find the sum (magnitude and direction) of the force vectors.
a. $\mathbf{A}+\mathbf{B}$

b. $\mathbf{A}+\mathbf{B}+\mathbf{C}$


## MOMENTUM: A DIFFERENT USE FOR VECTORS

We met the Principle of Conservation of Momentum in Y10, it says:

## Where there is no external force acting Momentum is Conserved

Where momentum is the vector quantity obtained from multiplying mass and velocity.

If you apply a bit of thought to what you remember from Y10 and then apply the vector ideas from above you should be able to imagine doing conservation of momentum calculations in two dimensions by resolving the velocity vectors into two perpendicular directions and applying the conservation law to each direction. You may well be required to do these calculations when covering the momentum topic. That, however, is not the point of this section.

While it is true that in Physics, and certainly in Engineering, vectors usually correspond to magnitudes and directions in two or three dimensional real space, they are not limited to only representing real space. In Maths you will meet Argand diagrams in which complex numbers are represented as vectors, but with one component being in the direction of imaginary numbers! That might seem extraordinarily abstract, but electrical engineers actually use Argand Diagrams as a mathematical tool to solve real problems involving resistors, capacitors and inductors.

In a 2017 paper Akihiro Ogura ${ }^{2}$ proposed using quite different vector diagrams as a way of illustrating momentum conservation in one dimension - i.e. the same as we did in Y10, everything happening in one straight line:

Example: Consider two objects colliding; Object A has a mass of 1 kg and a velocity before the collision of $4 \mathrm{~ms}^{-1}$ to the right. Object $B$ has a mass of 4 kg and a velocity of $0.25 \mathrm{~ms}^{-1}$ also to the right.

Obviously we would draw a diagram, and define going right as positive (in Physics the usual symbol for momentum is $p$ ):
$\xrightarrow{\text { Positive direction }}$

$p_{A}=4 \mathrm{kgms}^{-1}$


Clearly the total mass before the collision is 5 kg and the total momentum before the collision is $5 \mathrm{kgms}^{-1}$. Assuming no outside force acts the total momentum after the collision and the total mass will be unchanged. Ogura envisioned this in a vector space where one component was mass and the other was momentum:

[^1]

The vector $\vec{\varepsilon}$ is the vector sum of the two vectors representing the motion and mass of $A$ and $B$ before the collision, its mass component is 5 kg and its momentum component is $5 \mathrm{kgms}^{-1}$.

If the two bodies coalesced into one on collision the vector $\vec{\varepsilon}$ would represent the mass and motion of that body.

The velocity of the final body would be:
$v=\frac{p}{m}=\frac{5}{5}=1 \mathrm{~ms}^{-1}$
Two bodies in a collision sticking together to become one, but still obeying the conservation of momentum is known as a perfectly inelastic collision.

However, if we are told that A rebounds with a velocity of $-0.5 \mathrm{~ms}^{-1}$
(Velocities and momentums after the collision are indicated by ")



$p_{A}{ }^{\prime \prime}=-0.5 \mathrm{kgms}^{-1}$

$\xrightarrow{p_{B}{ }^{\prime \prime}=?}$


In Ogura's vector space the vector $\mathbf{B}^{\prime \prime}$ which represents the motion of $B$ after the collision is given by the vector subtraction of the vector $\mathbf{A}^{\prime \prime}$ from the total mass and momentum vector $\vec{\varepsilon}$

The vector $B^{\prime \prime}$ has the components $(4,4.5)$. The momentum of $B$ after the collision is therefore $4.5 \mathrm{kgms}^{-1}$ and the velocity is:

$$
v_{B}{ }^{\prime \prime}=\frac{p}{m}=\frac{4.5}{4}=1.125 \mathrm{~ms}^{-1}
$$

A difficult problem in collisions involves what are known as perfectly elastic collisions. In this case the kinetic energy before the collision is conserved as well as momentum being conserved (no energy is lost as heat and sound). Normally we solve elastic collision problems with simultaneous equations:

$$
m_{A} v_{A}+m_{B} v_{B}=m_{A} v_{A} "+m_{B} v_{B} "
$$

and

$$
1 / 2 m_{A}\left(v_{A}\right)^{2}+1 / 2 m_{B}\left(v_{B}\right)^{2}=1 / 2 m_{A}\left(v_{A} "\right)^{2}+1 / 2 m_{B}\left(v_{B} "\right)^{2}
$$

Two simultaneous equations allow us to find two unknowns. In other words if $v_{A}$ and $v_{B}$ are known then for an elastic collision $v_{A}$ " and $v_{B}$ " can both be found and only have one physically possible solution.

Although in principle this is straightforward simultaneous equations containing squares are not the nicest things and so solving these problems can be amongst the most difficult that we do at A level.

The beauty of Ogura's vector space is that for elastic collisions the changes to the momentum of each object are symmetrical around the total momentum vector in the diagram.

That is for a perfectly elastic collision, and for the points as marked on the grid, $B B^{\prime}=B^{\prime} B^{\prime \prime}$ and $A A^{\prime}=A^{\prime} A^{\prime \prime}$ :


Therefore, from the diagram, a perfectly elastic collision with the same starting conditions as before will have the final momentums and velocities:
$p_{A}{ }^{\prime \prime}=-2 \mathrm{kgms}^{-1}$ and $p_{B}{ }^{\prime \prime}=7 \mathrm{kgms}^{-1}$
$v_{A}{ }^{\prime \prime}=\frac{-2}{1}=-2 \mathrm{~ms}^{-1}$ and $v_{B}{ }^{\prime \prime}=\frac{7}{4}=1.75 \mathrm{~ms}^{-1}$
We can check that the kinetic energies before and after are indeed the same:
Before: $\quad 1 / 2 \times 1 \times 4^{2}+1 / 2 \times 4 \times 0.25^{2}=8.125$ J
After: $\quad 1 / 2 \times 1 \times(-2)^{2}+1 / 2 \times 4 \times 1.75^{2}=8.125 \mathrm{~J}$
Ogura therefore provides a straightforward graphical vector method for solving problems that are tough algebraically.

FINAL EXERCISE

1. On the grid on the next page construct the mass/momentum vectors for two objects; A target T , with a mass of 40 g , travelling left at $1.5 \mathrm{~ms}^{-1}$ and a projectile $P$ fired after it with a velocity of $56 \mathrm{~ms}^{-1}$ and a mass of 5 g . Sketch a diagram showing the relevant information about the two objects below.
Assuming that when the projectile strikes the target it is embedded into it, therefore producing a perfectly inelastic collision construct the mass/momentum vector for the combine object PT and calculate its velocity.

$V_{P T}=$ $\qquad$
2. Two objects $A$ and $B$ are travelling towards one another. $A$ is coming from the left, has a mass of 3 kg and a velocity of $2 \mathrm{~ms}^{-1}, \mathrm{~B}$, coming from the right, is travelling at $0.5 \mathrm{~ms}^{-1}$ and has a mass of 6 kg .
They collide perfectly elastically; on the grid below construct the mass/momentum vectors for this collision to find $v_{A}{ }^{\prime \prime}$ and $v_{B}{ }^{\prime \prime}$ (include directions). Show that the kinetic energy before and after the collision is the same.

$\qquad$
$v_{B}{ }^{\prime \prime}$
$K E_{\text {before }}$ $\qquad$

[^0]:    ${ }^{1}$ http://iblog.dearbornschools.org/singley/wp-content/uploads/sites/781/2014/06/component-vector-practice.pdf

[^1]:    ${ }^{2}$ http://iopscience.iop.org/article/10.1088/1361-
    6404/aa750b/meta;jsessionid=12579AB56430E57B76FF35663AB92DA9.c4.iopscience.cld.iop.org

